Effective Teaching of Decimals: Evaluating Teachers' Practices

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This study arose from a request from teachers for help in evaluating their current methods of teaching decimals. Results reported here are from 14 classes of students, aged 9 through 12, and demonstrate that it is possible to find marked differences in progress of different classes. Students were given a common pre-test and post-test. These results were related to the models and reported teaching procedures used. The main factors leading to students' improvement appeared to be careful planing to meet their needs based on teachers' knowledge of the underlying concepts, the use of a clear model that students could use to visualise decimal division, and careful bridging from visualisation to numerical forms.

There is often a mismatch between educational research that advances theoretical understanding of learning and the day-to-day needs of teachers to know what they are doing effectively. This issue has been emphasised by Viviane Robinson in her book, Problem-Based Methodology (Robinson, 1993). In this book she speaks of dedicating her work to:

> all those educational researchers who have wished that educational practitioners and policy-makers would take more notice of their work, and to all those practitioners and policy-makers who have wondered when educational researchers would produce something that was worth taking notice of. (p. vii)

This comment strikes a responsive chord with both teachers and researchers. The methodology that she proposes is an intensive one, in which researchers and teachers frequently check their understanding of one another's needs and assumptions.

The study reported here arose from a similar concern, although it used briefer consultations than those proposed by Robinson. It started with a request from a principal to help the teachers at her school assess which of the various ways of teaching decimals that they were currently using were most effective. We knew that this was often a difficult topic for students, despite its logical nature. We knew that many different ways were used to introduce this topic to students, yet students continue to have misconceptions in this domain.

Background

Gelman (1999) discussed the fact that all individuals, schooled or unschooled, can carry out additive processes, but that a cognitive shift was needed to understand multiplicative processes. Multiplicative processes include understanding the multiplicative nature of the place value system, the divisions necessary for understanding decimal fractions, common fractions and ratio. These are thought to be the most difficult concepts taught in primary school.

There are several aspects of decimal fractions which students find difficult. They are difficult because they do not follow the rules, or schema, that students have developed for whole numbers, nor are they the mirror image of this system reflected at the decimal point (see for example, Baturo, 1997; Irwin, 1997; Stacey & Steine, 1998). If students try to generalise from their understanding of whole numbers, they are likely to be incorrect. They must appreciate that decimal fractions are parts of a whole, and that the numbers that are used for counting can be subdivided. They need to understand the relationship of decimal fractions and the language used to describe them, in that the unit that a decimal fraction is a proportion of is always implied, not stated. They need to understand the multiplicative character of the number system, perhaps for the first time, in which the value or a number in each place is 10 times what it would be if it were in a place to the right and vice versa. This is different from the way in which whole numbers may be understood using an overflow image, in which each place gets filled up when you get to 9 (or 99 or 999), and then you use another column to the left to write the next number.

A wealth of students' misconceptions can be traced to their failure to understand these basic qualities of decimals. For example, it is common for students to believe that there is a "oneths" column to the right of the decimal point. They may believe that adding another tenth to 0.9 yields 0.10. They may believe that 4 hundredths is written as 0.400. They may believe that decimal fractions work in a manner that is a mirror image of whole numbers, so that 0.0023 is larger than 0.23. All of these misconceptions are common. They all reflect an imperfect understanding of what a decimal fraction is and how it is shown in writing.

Several methods for helping students understand the basic nature of decimal fractions have been demonstrated to be effective in comparative studies by educational researchers. Some emphasise a strong visual image, while others emphasise conflict between students' existing belief and a known example.

Swan (1983, 1990) taught two parallel classes of 12-13 year-old students using different methods. In both classes he provided students with a number-line model for understanding the meaning of decimal notation, and encouraged students to visualise this when doing simple addition and subtraction or comparing the size of numbers that included decimal fractions. One class was taught by what he called a "positive only teaching style". In this class concepts were explained and methods for obtaining correct answers taught using the number line. Students then practised what they had learned on similar problems. The other class was taught using a "conflict teaching style". This teaching had four phases. Initially, students were given problems that were likely to expose their misconceptions. Next they were asked to repeat the problems using the number line model that had been shown to the other group. This was followed by a class discussion of the errors and misconceptions exposed by the contrasting results of the first two phases. Finally there was a consolidation phase in which students practiced similar problems, as had the other class. The group taught by the conflict method was reported to be more difficult to teach because of the extent of their debates, but the students made significantly more gain between pre-test and post-test than did the other class.

Another study that provided a method for teaching decimal fractions was that of Wearne and Hiebert (e. g. Hiebert & Wearne, 1989; Wearne 1990). Their focus was also on students' conceptual understanding. In laboratory and classroom studies of fourth, fifth and sixth grade students in the United States, they introduced decimals through the use of place value blocks. The large cube was used to represent a unit, the flat block to represent a tenth of this, a long block to represent a hundredth of the unit, and the small cube to represent a thousandth of the large cube. They found that students were able to show markedly greater understanding of numerical representation of decimal fractions after instruction using this concrete model. For example, in one study of fourth grade students, 20% of the students gave accurate answers on interviews before instruction, and eight weeks after the instruction period 93% of responses were correct on measures using the materials and 75% of responses on a transfer task with numbers only were correct (Wearne & Hiebert, 1989).

A third method, useful for helping students overcome their misconceptions, was developed by Irwin (e.g., 1997). This program used students' everyday knowledge to help them overcome misconceptions. Students worked in pairs to solve a number of problems that involved slightly unusual forms of decimals yet in familiar contexts. For example, they were asked to add the cost of $4.95 and 91.9 cents for a particular outing, or how many Samoan tala they would get for $NZ10 if the exchange rate was $NZ1 to $S1.5989. In these cases, application of simplistic rules that they had learned for handling whole numbers like "adding a 0" to multiply by 10, or "lining up the decimal point" did not work, and their knowledge of the setting made them realise that their schemas must be modified. Students who worked through these cognitive conflicts made significantly more progress on a post-test than did students who worked on similar problems without contexts.

Other important ways of improving instruction have been developed by researchers, including the interactive computer program of Stacey and diagrams of Baturo. Aspects of all of these methods are used in schools in New Zealand, although many teachers will not be familiar with the underlying research. There are several other methods used that are recommended in textbooks or teaching guides. However, while educational research focuses on the value of single teaching procedures, classrooms usually use a wide variety of procedures. They rarely provide careful comparison studies like those in the cases given above. While researchers want to demonstrate the value of a particular method, teachers change their methods or models frequently, according to what they perceive as the needs of their students.

This study attempted to capture the wealth of procedures used in classrooms by teachers who were not restrained by the need to prove the value of a particular method, but who still wanted to know if their teaching was effective.

Method

Participants

The data presented below came from 14 classes in four schools that are members of the University of Auckland Consortium of Schools. They are in the West, South, and Central parts of Auckland, and covered a range of economic backgrounds, from Decile 1 (low family income) to Decile 7 (moderate family income). The students represented the cross-section of ethnic groups in this city: Maori; Pacific nations; Pakeha/European; Indian; Asian; and others. The classes were all composite Year 5 and 6 (ages 9 and 10) or composite. Year 7 and 8 classes (ages 11 and 12). Some classes were cross-grouped for mathematics, making them relatively homogeneous in achievement, and others were heterogeneous in attainment.

The teachers varied in their ethnicity, place of teacher education, and years of experience, although all had been teaching for at least three years.

Procedure

The procedure for evaluating teaching effectiveness and class progress was negotiated with the teachers. It was decided that all teachers would give the same test as a pre-test and a post-test. This was based on the Chelsea Diagnostic Test of Place Value and Decimals (Hart et al., 1985), but did not include all of the items. This test has levels that are described as: (0) little understanding of whole number values; (1) understanding of whole numbers; (2) understanding of tenths; (3) understanding hundredths and thousandths; (4) the relation of value to adjacent places; (5) more complex relationships between places; and (6) decimals as the result of division and as infinitely divisible. The selection of items used yielded a maximum raw score of 37, leaving out one scored item from each of Levels 5 and 6 of the original test. This test is intended to be diagnostic, and presents several different models for representing decimals, if the concept is understood. The ordering of levels on the original test was pragmatic, reflecting the development of students in the English sample. Because this assessment started with understanding of whole numbers, it was suitable for students who still had a weak grasp of whole numbers as well as those with more advanced skills of decimals.

Teachers taught a unit on this topic that was usually of three weeks' duration, with days lost for other activities. Teachers decided what and how they would teach, but they agreed to keep records of their planning, the changes they made, samples of students' work and any homework. Some teachers planned together and others planned individually. At the end of the teaching unit each teacher was interviewed.

Interview questions covered the selected achievement objectives and learning outcomes, use made of information from the pre-test, the main models, resources and activities used to teach this unit, what they thought were the most difficult aspects of decimals for students to grasp, and questions about the teachers' satisfaction with the unit, their perception of the students's learning, and their confidence in teaching this unit.

This report concentrates on the progress made by the students, and provides snapshots of the teaching provided for the classes of those that made most and least progress.

Results

Every teacher taught differently, even those who had planned together. They all used different resources although they sometimes obtained these resources from common sources. They taught the unit for between 2 and 4 weeks, with most teaching for three weeks. In every case, mathematics was not held on some of the days because of other school events. Thus the time devoted to the unit varied between 8 days and 15 days. There was no relationship between the number of days for which the subject was taught and the results for the class.

All but one teacher used textbooks and worksheets, but nine different texts were used. Models included dividing chocolate and carrots to demonstrate tenths, using place value blocks, drawing area diagrams and number lines, using calculators for discovery, and games for practice. Some teachers saw themselves as the main resource used. Half of the teachers reported feeling confident about teaching decimals, with the less confident teachers usually teaching the classes with lower pre-test scores.

Every class progressed in average level on the test when assessed on the post-test, as did the vast majority of students. Progress was noted in the mean increase in level of understanding of this test, the mean increase in raw score, and in the percent of students who increased at least one of the Chelsea levels on the test (e.g., from understanding whole numbers to understanding tenths). Since classes started with different mean levels of understanding, the results have been divide into three groups: those with relatively low, medium or relatively high initial understanding.

Average Attainment and Improvement of Students in Classes Assessed.

In three classes, the majority of students failed to show an adequate understanding of whole numbers on the pre-test. The class with the highest percent of students moving up a level also had the highest mean increase in level and the highest increase in raw score.

Class B in this group made the most progress. Their teacher chose to teach only whole numbers and tenths. She used models that enabled students to visualise tenths, using chocolate that was divided, the number line, and moving on to grids and calculator work with fraction equivalents (1/2 = 0.5). She did very little exploration of decimal numbers without

Table 1
Classes that Started with a Low Level of Understanding of Place Value

Class (Years)	Initial Mean Level	Mean increase in level	Mean increase in raw score	Percent of students increasing by 1 or more levels
A (5/6)	0.14	0.34	1.23	23%
B (5/6)	0.20	0.84	6.40	84%
C (5/6)	0.32	0.77	5.27	60%

models that demonstrated their meaning. This teacher did not feel confident about teaching this topic at more complex levels but, by keeping instruction at a level suitable for the students, advanced their understanding.

The teacher whose class made the least progress (A) appeared to have pitched teaching at too high a level for these students. He was confident in teaching more advanced students and was still coming to grips with the understanding of these students.

The seven classes in Table 2 started with an initial understanding of whole numbers, but limited understanding of decimal fractions.

Table 2
Classes that Started with an Understanding of Whole Numbers but were not Strong on any Decimals

Class (Years)	Initial Mean Level	Mean increase in Levels	Mean increase in raw score	Percent of students increasing by 1 or more levels
D (5/6)	0.55	1.55	9.83	79%
E (5/6)	0.75	0.04	1.75	08%
F (7/8)	0.75	0.55	2.70	45%
G (5/6)	0.79	1.58	8.58	76%
H (5/6)	0.94	1.91	9.38	75%
I (5/6)	1.00	1.56	9.38	75%
J (7/8)	1.12	3.94	23.00	100%

These classes ranged from making little progress on this test, to making a great deal of progress, with the majority making substantial progress. This suggests that students with an initial grasp of whole numbers (at Level 1) are in a better position to make major gains than are students without this grasp. Class J made the most remarkable progress, with the majority of the class moving from Level 1 (understanding of whole numbers) to Level 5 (understanding of the relationship of places in decimals, multiplying by 10 and 100, etc). These were older students whose initial understanding was relatively uniform. The teacher's main purpose was to give his students a visual representation that would enable them to understand decimals and operate with them. The model that he emphasised was the number line. He used this model in class every day, starting classes with whole-class lessons on the placement of decimal numbers on a number line. For example, he would ask students to place a number between 3.45 and 3.46, or he would ask them to put some numbers, with a different number of decimal places, around a number like 1.8. He made up worksheets that gave students practice in thinking about decimals as places on an infinitely divisible number line. He also covered other activities, including transfer from the model to numerical form and adding and multiplying decimals, but students were expected to think about these activities in relationship to this one major visualisation. Their post-tests showed that they had incorporated this visualisation, but used their own words to write about it. Main factors that contributed to their success were likely to have included his thoughtful use of errors on the pre-tests, his own pedagogical content knowledge (he has a Diploma in Mathematics Education), his careful preparation, his strong emphasis on a visual model that had infinite divisions, and use of this model to solve numerical problems.

The teacher of class H had a class that was far more diverse initially, a fact that is masked by using class means. She had taught the older students in the previous year, and they scored between Level 1 and 4 on the pre-test. The younger students were new to her, most scored at Level 0. Her aims were to assure that students understood the difference between parts of numbers and whole numbers and had a firm understanding of the multiplicative nature of place value. Initially she taught the Year 5 and 6 students separately, explaining the difference between wholes and parts to the Year 5 students using area models drawn on the white board. As the unit progressed, she taught the class as a whole although exercises were sometimes different for the two groups. Her emphasis was always on the mathematical concepts behind numeration. She talked of the decimal point as a fence that separated whole numbers from fractional numbers and that multiplying or dividing by 10 or 100 could involve "jumping over that fence", not "moving the decimal point". The post-test results for the younger students showed that in addition to moving up 1, 2, or 3 levels on the test they also understood some items at the highest levels. For example, none of them made errors when asked how to write eleven tenths. She took advantage of the apprenticeship model in which the students who were less advanced learned from their more competent classmates (e.g., Rogoff, 1990).

The teacher of class E, which made the least progress, was the one who taught the unit for the longest period. She appeared to have taught too wide a range of concepts, and used a wide variety of models and resources, without enabling the students to consolidate their understanding. She was not confident in this topic, while the teachers of the two classes discussed above were both confident.

The four classes in Table 3 started with a general understanding of at least tenths. Three of these were classes of older students, and one was a selected group of younger students with above average attainment. In one of these classes most students already understood more complex relations between places in decimals.

Interesting teaching happened in Class K, which started with a full range of scores, from 0 to 6, and nearly all students moved up at least one level. Their teacher followed the general framework for moving from concrete models to diagrams to numbers. The teacher of class N had an accelerate class most of whom scored at level 5 initially. He did not specifically teach decimals, but integrated the topic into ongoing work. The post-tests of many students looked as though they had been taken in a rush, without much care, an impression confirmed by this teacher.

Table 3

Classes that Started with a Higher Average Understanding of Decimals

Class (Years)	Initial Mean Level	Mean increase in Levels	Mean increase in raw score	Percent of students increasing by 1 or more levels
K (high 5/ 6)	1.60	1.36	7.12	82%
L (7/8)	1.73	1.5	7.70	75%
M (7/8)	1.79	1.08	7.90	73%
N (high 7/8)	3.89	0.55	-0.82	45%

Summary and Discussion

All of these teachers taught from a common, mandatory curriculum yet their teaching methods varied widely. They drew on a broad range of models and resources, as is typical of current practice. No two classes were taught in the same way. Even those teachers who had planned jointly used different models, resources, and activities. However, there were similarities among the most successful teachers.

Those teachers whose students made the most progress in this unit shared several characteristics. (1) They thought carefully about what their students needed to learn; (2) they planned carefully for their students' needs; (3) they used a model which enabled students to visualise what decimals were; and (4) they were careful in bridging from this visualised model to the numerical form. They reported making the topic fun, and expected their students to work, for example, through doing homework. On the other hand, none of these most successful teachers put heavy emphasis on place value columns, as do many teachers in New Zealand schools.

Most of the teachers whose students made good or outstanding progress had a good understanding of place value as a multiplicative concept, although they did not speak of it this way. They were familiar with the difficulties that students often have in this domain. Usually a teacher's confidence in an area goes along with competent teaching. It is interesting that the teacher of class B was an exception in this regard, although she considered herself to be more confident after her successful teaching. She thought carefully about what her less advanced students needed to know next, and by limiting her teaching to this, enabled her class to progress. Other less confident teachers all appeared to have attempted to teach too much, using too many models. It is a good reminder that one can be a good teacher of primary school mathematics without being a confident mathematician, as long as one thinks carefully about what needs to be learned and make sure to know that part well.

Meetings were held with the teachers in each school to report back on the findings of the research. Teachers reported finding these meeting very constructive. Some group problem solving went on, as teachers whose classes had had varying levels of success talked about improving their practice. They reported that the evaluation had met their needs.

Although no teacher mentioned any of the educational research on this topic, several taught in ways that incorporated aspects of this research. No teacher mentioned using cognitive conflict, but their comments on students' usual difficulties and the methods that they used suggested that they did understand students' confusions and the need to modify their schemas developed for whole numbers. Several mentioned the use of place value blocks for presenting the relationship of wholes, tenths, hundredths, and thousandths. However, none of the teachers who used this model found it particularly helpful. They found that the students were confused by the fact that all blocks were said to represent different values from those used in learning about whole numbers. Teacher J, who made more extensive use of the number, appeared to have used it primarily in Swan's sense of "positive only teaching style" (Swan, 1990). The practice of the successful teachers did confirm the importance of visualisation (e.g., Presmeg, 1997), solid pedagogical content knowledge (Schulman, 1986) and learning through apprenticeship (Rogoff, 1990).

What this research has done is identify some teachers who have been very successful in teaching a difficult topic. They have done so in conditions that are more complex than those of much educational research. They lead us to ask if the characteristics that have been identified from these classes would also be identified in a more regulated study. It is through this interplay of teachers' practice and researchers' findings that a more complete understanding of teaching and learning is likely to occur.

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